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Pileup Probabilities and Events per Bunch-Crossing

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Abstract

This note collects some formulae and numbers for dealing with pileup at the LHC.

1 Introduction

The total number of events per bunch-crossing follows a Poisson distribution

$$P(N_t) = \frac{\nu_t^{N_t}}{N_t!} \mathrm{e}^{-\nu_t} \tag{1}$$

with mean

$$\nu_t = \langle N_t \rangle = \sigma_t \, \mathcal{L} \, \langle \Delta t_{bunch} \rangle \, , \tag{2}$$

where the mean bunch distance is given by

$$\langle \Delta t_{bunch} \rangle = \frac{1}{f_{LHC} \, k} \, . \tag{3}$$

The LHC revolution frequency is $f_{LHC} = 11.245\,\mathrm{kHz}$. For the total cross-section we assume $\sigma_t = 110\,\mathrm{mb}$. The luminosity $\mathcal{L}$ and the number of bunches k depend on the running conditions. We consider two cases:

1. **A typical $\beta^* = 0.5\,\mathrm{m}$ scenario: $\mathcal{L} = 10^{33}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ and $k = 2808$ ($\mathcal{L}/k = 3.6 \times 10^{29}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$):**
   In this case, $\langle \Delta t_{bunch} \rangle = 31.67\,\mathrm{ns}$ (not the minimum bunch distance of 25 ns!). This yields $\nu_t = 3.5$ events per bunch crossing.

   Some distribution values for $P(N_t)$ are given in Table 1. We conclude that the probability of observing at least 1 event is $1 - 0.03 = 97\,\%$, and the pileup probability $P(N_t \geq 2) = 86.4\,\%$.

n	0	1	2	3	4	5
$P(N_t = n)$	0.030	0.106	0.185	0.216	0.189	0.132
$P(N_t > n)$	0.970	0.864	0.679	0.463	0.274	0.142

Table 1: Poisson distribution for the total number of events per bunch-crossing for $\mathcal{L}/k = 3.6 \times 10^{29}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 3.5$).

2. **The typical $\beta^* = 90\,\mathrm{m}$ scenario: $\mathcal{L} = 3 \times 10^{30}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ and $\mathbf{k} = 156$ ($\mathcal{L}/\mathbf{k} = 1.9 \times 10^{28}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$):**
   Here, the mean bunch distance is $\langle \Delta t_{bunch} \rangle = 570\,\mathrm{ns}$, and per bunch crossing a mean number of $\nu_t = 0.188$ events is observed. Table 2 gives some values for $P(N_t)$. In this scenario, the probability of observing at least 1 event is 17 %, and the pileup probability $P(N_t \geq 2) = 1.5\,\%$.

n	0	1	2	3
$P(N_t = n)$	0.8285	0.1558	0.0147	0.0009
$P(N_t > n)$	0.1715	0.0157	0.0010	0.0001

Table 2: Poisson distribution for the total number of events per bunch-crossing for $\mathcal{L}/k = 1.9 \times 10^{28}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 0.188$).

2 Pileup Probability for Specific Event Classes

Question: What is the probability of observing in 1 bunch-crossing a total number of N_t events out of which N_A belong to class A, N_B to class B etc.?
We decompose the total cross-section as

$$\sigma_t = \sigma_A + \sigma_B + ... + \sigma_R \tag{4}$$

or, in terms of the number of events,

$$N_t = N_A + N_B + ... + N_R \tag{5}$$

where the event class R stands for the unspecified rest. In this note, we consider the two simplest cases with 1 and 2 specified event classes.

2.1 One Specified Event Class

The simplified question is: What is the probability of observing in 1 bunch-crossing a total number of N_t events out of which N_A are of type A, for example elastic scattering events? The event number decomposition now reads:

$$N_t = N_A + N_R \tag{6}$$

The quantity requested is the two-dimensional probability $P(N_A, N_t)$. From the independence of the classes A and R follows directly:

$$P(N_A, N_t) = \frac{\nu_A^{N_A}}{N_A!}\,\mathrm{e}^{-\nu_A}\,\frac{(\nu_t - \nu_A)^{N_t - N_A}}{(N_t - N_A)!}\,\mathrm{e}^{-(\nu_t - \nu_A)} \tag{7}$$

A different approach – which formally confirms the independence of A and R – is to express $P(N_A, N_t)$ as

$$P(N_A, N_t) = P(N_A|N_t)\,P(N_t) \tag{8}$$

where $P(N_t)$ is given by (1), and the conditional probability $P(N_A|N_t)$ follows the binomial distribution

$$P(N_A|N_t) = \left(\begin{array}{c} N_t \\ N_A \end{array} \right) \left(\frac{\nu_A}{\nu_t}\right)^{N_A} \left[1 - \frac{\nu_A}{\nu_t}\right]^{N_t - N_A} \tag{9}$$

Note that here the prerequisites for a Poisson approximation are not fulfilled: N_t is not very big, and ν_A/ν_t is not necessarily very small (e.g. it isn't if A stands for elastic scattering).
Combining (1) and (9) in (8) we obtain

$$P(N_A, N_t) = \left(\begin{array}{c} N_t \\ N_A \end{array} \right) \left(\frac{\nu_A}{\nu_t} \right)^{N_A} \left[1 - \frac{\nu_A}{\nu_t} \right]^{N_t - N_A} \frac{\nu_t^{N_t}}{N_t!} \mathrm{e}^{-\nu_t} \tag{10}$$

which is equivalent to (7).
As examples, Tables 3 and 4 show $P(N_A, N_t)$ for the typical 0.5 m and 90 m running scenarios, in the case where A stands for elastic scattering.

	N_t						
N_A	0	1	2	3	4	5	$P(N_A)$
0	0.030	0.077	0.098	0.083	0.053	0.027	0.387
1	0	0.029	0.073	0.093	0.080	0.051	0.368
2	0	0	0.014	0.035	0.045	0.038	0.175
3	0	0	0	0.0044	0.011	0.014	0.055
4	0	0	0	0	0.001	0.003	0.013

Table 3: $P(N_A, N_t)$ for A = elastic assuming $\sigma_{elastic} = 30\,\mathrm{mb}$, for $\mathcal{L}/k = 3.6 \times 10^{29}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 3.5$, $\nu_A = 0.95$, $\nu_A/\nu_t = 0.273$, $1 - \nu_A/\nu_t = 0.727$).

	N_t				
N_A	0	1	2	3	$P(N_A)$
0	0.8285	0.1133	0.0077	0.00035	0.9499
1	0	0.0426	0.0058	0.00040	0.0488
2	0	0	0.0011	0.00015	0.0013
3	0	0	0	1.8×10^{-5}	2.1×10^{-5}

Table 4: $P(N_A, N_t)$ for A = elastic assuming $\sigma_{elastic} = 30\,\mathrm{mb}$, for $\mathcal{L}/k = 1.9 \times 10^{28}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 0.188$, $\nu_A = 0.051$, $\nu_A/\nu_t = 0.273$, $1 - \nu_A/\nu_t = 0.727$).

Special limit:
Which is the probability of observing N_t events out of which 1 is of a **very rare** type A, i.e. $N_A = 1$ and $\nu_A/\nu_t \to 0$?
Eqn. (9) now reduces to

$$P(N_A = 1|N_t) = N_t \frac{\nu_A}{\nu_t} \left[1 - \frac{\nu_A}{\nu_t} \right]^{N_t - 1} \to N_t \frac{\nu_A}{\nu_t}\,, \tag{11}$$

and our result is

$$P(N_A = 1, N_t) \to N_t \frac{\nu_A}{\nu_t} \frac{\nu_t^{N_t}}{N_t!} \mathrm{e}^{-\nu_t} = \nu_A \frac{\nu_t^{N_t - 1}}{(N_t - 1)!} \mathrm{e}^{-\nu_t} \tag{12}$$

2.2 Two Specified Event Classes

What is the probability of observing in 1 bunch-crossing a total number of N_t events out of which N_A belong to class A and N_B to class B?

Analogously to the previous section, $P(N_A, N_B, N_t)$ can be written either as

$$P(N_A, N_B, N_t) = \frac{\nu_A^{N_A}}{N_A!}\,\mathrm{e}^{-\nu_A}\,\frac{\nu_B^{N_B}}{N_B!}\,\mathrm{e}^{-\nu_B}\,\frac{(\nu_t-\nu_A-\nu_B)^{N_t-N_A-N_B}}{(N_t-N_A-N_B)!}\,\mathrm{e}^{-(\nu_t-\nu_A-\nu_B)} \tag{13}$$

or as

$$\begin{aligned} P(N_A, N_B, N_t) &= \binom{N_A+N_B}{N_A}\left(\frac{\nu_A}{\nu_A+\nu_B}\right)^{N_A}\left[1-\frac{\nu_A}{\nu_A+\nu_B}\right]^{N_B} \\ &\times \binom{N_t}{N_A+N_B}\left(\frac{\nu_A+\nu_B}{\nu_t}\right)^{N_A+N_B}\left[1-\frac{\nu_A+\nu_B}{\nu_t}\right]^{N_t-N_A-N_B} \\ &\times \frac{\nu_t^{N_t}}{N_t!}\mathrm{e}^{-\nu_t} \end{aligned} \tag{14}$$

Tables 5 and 6 show the case of A = elastic and B = NSD for the two scenarios considered.

	N_t					
N_A, N_B	0	1	2	3	4	5
0, 0	0.030	0.010	0.002	0.0002	2×10^{-5}	8×10^{-7}
0, 1	0	0.067	0.022	0.004	0.0004	4×10^{-5}
1, 0	0	0.029	0.010	0.002	0.0002	2×10^{-5}
0, 2	0	0	0.075	0.025	0.004	0.0004
1, 1	0	0	0.064	0.021	0.003	0.0004
2, 0	0	0	0.014	0.005	0.0007	8×10^{-5}
0, 3	0	0	0	0.055	0.018	0.003
1, 2	0	0	0	0.071	0.023	0.004
2, 1	0	0	0	0.030	0.010	0.002
3, 0	0	0	0	0.004	0.001	0.0002
0, 4	0	0	0	0	0.031	0.010
1, 3	0	0	0	0	0.053	0.017
2, 2	0	0	0	0	0.034	0.011
3, 1	0	0	0	0	0.010	0.003
4, 0	0	0	0	0	0.001	0.0003

Table 5: $P(N_A, N_B, N_t)$ for A = elastic, B = NSD, (i.e. R = SD) assuming $\sigma_{elastic} = 30\,\mathrm{mb}$ and $\sigma_{NSD} = 70\,\mathrm{mb}$, for $\mathcal{L}/k = 3.6 \times 10^{29}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 3.5$, $\nu_{elastic} = 0.95$, $\nu_{NSD} = 2.22$, $\nu_{SD} = 0.33$).

	N_t			
N_A, N_B	0	1	2	3
0, 0	0.8285	0.0140	0.00012	7×10^{-7}
0, 1	0	0.0993	0.0017	0.00001
1, 0	0	0.0425	0.00072	6×10^{-6}
0, 2	0	0	0.0059	0.00010
1, 1	0	0	0.0051	0.00009
2, 0	0	0	0.0011	0.00002
0, 3	0	0	0	0.00024
1, 2	0	0	0	0.00031
2, 1	0	0	0	0.00013
3, 0	0	0	0	0.00002

Table 6: $P(N_A, N_B, N_t)$ for A = elastic, B = NSD, (i.e. R = SD) assuming $\sigma_{elastic} = 30\,\mathrm{mb}$ and $\sigma_{NSD} = 70\,\mathrm{mb}$, for $\mathcal{L}/k = 1.9 \times 10^{28}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 0.188$, $\nu_{elastic} = 0.051$, $\nu_{NSD} = 0.120$, $\nu_{SD} = 0.017$).

3 Number of Events per Bunch-Crossing under the Condition of Having a Trigger

Suppose we trigger on events of class A (which may even represent a minimum bias trigger as a special case). In one bunch crossing there can be at most one trigger, even if several events of type A have occurred. Hence, the trigger condition can be written as $N_A \geq 1$. Which is the probability distribution and which is the mean of the total number of events in a bunch crossing under the condition that a trigger has been given?

3.1 Probability Distribution

The conditional probability distribution we are aiming at is formally expressed as

$$P(N_t | N_A \geq 1) = \frac{P(N_t, N_A \geq 1)}{P(N_A \geq 1)} \tag{15}$$

Since $P(N_A)$ follows a Poisson distribution, the denominator becomes

$$P(N_A \geq 1) = 1 - \mathrm{e}^{-\nu_A} \tag{16}$$

where as the enumerator is resolved using (8) and (9):

$$\begin{aligned} P(N_t, N_A \geq 1) &= P(N_t) \sum_{N_A=1}^{N_t} \binom{N_t}{N_A} \left(\frac{\nu_A}{\nu_t}\right)^{N_A} \left[1 - \frac{\nu_A}{\nu_t}\right]^{N_t - N_A} \\ &= P(N_t) \left[\sum_{N_A=0}^{N_t} \binom{N_t}{N_A} \left(\frac{\nu_A}{\nu_t}\right)^{N_A} \left(1 - \frac{\nu_A}{\nu_t}\right)^{N_t - N_A} - \left(1 - \frac{\nu_A}{\nu_t}\right)^{N_t} \right] \\ &= P(N_t) \left[1 - \left(1 - \frac{\nu_A}{\nu_t}\right)^{N_t} \right] \end{aligned} \tag{17}$$

Finally

$$\begin{aligned} P(N_t|N_A \geq 1) &= P(N_t)\frac{1-\left(1-\frac{\nu_A}{\nu_t}\right)^{N_t}}{1-\mathrm{e}^{-\nu_A}} \\ &= \frac{\nu_t^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t}\,\frac{1-\left(1-\frac{\nu_A}{\nu_t}\right)^{N_t}}{1-\mathrm{e}^{-\nu_A}} \qquad (18) \end{aligned}$$

Let us now consider two special cases:

- **Minimum Bias Trigger:**
  In this case, $\nu_A = \nu_t$. Hence

$$P(N_t|N_t \geq 1) = \frac{\nu_t^{N_t}}{N_t!}\,\frac{\mathrm{e}^{-\nu_t}}{1-\mathrm{e}^{-\nu_t}} \qquad (19)$$

- **Triggering on a very rare event class:**
  In this limit, $\nu_A \ll 1$.

$$\begin{aligned} P(N_t|N_A \geq 1) &\to \frac{\nu_t^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t}\,\frac{1-\left(1-N_t\frac{\nu_A}{\nu_t}\right)}{1-1+\nu_A} \\ &= \frac{\nu_t^{N_t-1}}{(N_t-1)!}\,\mathrm{e}^{-\nu_t} \\ &= P(N_t-1) \qquad (20) \end{aligned}$$

3.2 Mean Number of Events

The mean number of events in a bunch crossing where a trigger has been observed is given by

$$\langle N_t\rangle_{N_A\geq 1} = \sum_{N_t=0}^{\infty} N_t\, P(N_t|N_A \geq 1) \qquad (21)$$

Using (18) for the general case gives

$$\begin{aligned} \langle N_t\rangle_{N_A\geq 1} &= \frac{\displaystyle\sum_{N_t=0}^{\infty} N_t\,\frac{\nu_t^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t}\left[1-\left(1-\frac{\nu_A}{\nu_t}\right)^{N_t}\right]}{1-\mathrm{e}^{-\nu_A}} \\ &= \nu_t\,\frac{\displaystyle\sum_{N_t=1}^{\infty} \frac{\nu_t^{N_t-1}}{(N_t-1)!}\,\mathrm{e}^{-\nu_t}\left[1-\left(1-\frac{\nu_A}{\nu_t}\right)^{N_t-1}\left(1-\frac{\nu_A}{\nu_t}\right)\right]}{1-\mathrm{e}^{-\nu_A}} \\ &= \nu_t\,\frac{\displaystyle\sum_{N_t=0}^{\infty} \frac{\nu_t^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t} - \left(1-\frac{\nu_A}{\nu_t}\right)\sum_{N_t=0}^{\infty} \frac{\nu_t^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t}\left(1-\frac{\nu_A}{\nu_t}\right)^{N_t}}{1-\mathrm{e}^{-\nu_A}} \\ &= \nu_t\,\frac{1-\left(1-\frac{\nu_A}{\nu_t}\right)\mathrm{e}^{-\nu_A}\displaystyle\sum_{N_t=0}^{\infty}\frac{\left[\nu_t\left(1-\frac{\nu_A}{\nu_t}\right)\right]^{N_t}}{N_t!}\,\mathrm{e}^{-\nu_t\left(1-\frac{\nu_A}{\nu_t}\right)}}{1-\mathrm{e}^{-\nu_A}} \\ &= \nu_t\,\frac{1-\left(1-\frac{\nu_A}{\nu_t}\right)\mathrm{e}^{-\nu_A}}{1-\mathrm{e}^{-\nu_A}} \\ &= \nu_t\left[1+\frac{\nu_A}{\nu_t}\,\frac{\mathrm{e}^{-\nu_A}}{1-\mathrm{e}^{-\nu_A}}\right] \qquad (22) \end{aligned}$$

Again, we shall illustrate this formula with the same special cases as in the previous section.

- **Minimum Bias Trigger:**
  In this case, $\nu_A = \nu_t$. Hence

$$\langle N_t \rangle_{N_t \geq 1} = \nu_t \left[1 + \frac{\mathrm{e}^{-\nu_t}}{1 - \mathrm{e}^{-\nu_t}} \right] \tag{23}$$

  For $\nu_t = 3.5$, we obtain

$$\langle N_t \rangle_{N_t \geq 1} = 3.5 \left(1 + \frac{0.030}{0.970} \right) = 3.6 \,,$$

  and for $\nu_t = 0.188$:

$$\langle N_t \rangle_{N_t \geq 1} = 0.188 \left(1 + \frac{0.829}{0.171} \right) = 1.097 \,.$$

- **Triggering on a very rare event class:**
  In this limit, $\nu_A \ll 1$.

$$\begin{aligned} \langle N_t \rangle_{N_A \geq 1} &\rightarrow \nu_t \left[1 + \frac{\nu_A}{\nu_t} \frac{1 - \nu_A}{1 - 1 + \nu_A} \right] \\ &= \nu_t + 1 - \nu_A \qquad (24) \\ &\rightarrow \nu_t + 1 \qquad (25) \end{aligned}$$

  Note that this formula – "very well known to everybody for at least 30 years" [1] – applies only to this special limit and not to the general case.

Figure 1 shows $\langle N_t \rangle_{N_A \geq 1}$ as a function of ν_A for $\nu_t = 3.5$ in the general case (22) and in the approximation for small ν_A (24). Note that in the cases of elastic scattering ($\nu_A = 0.95$) and single diffraction ($\nu_A = 0.33$) the general formula must be used whereas for Double Pomeron exchange ($\nu_A = 0.03$ assuming $\sigma_{DPE} = 1\,\mathrm{mb}$) both approximations (24) and (25) are within $\pm 0.33\,\%$ from the true value.

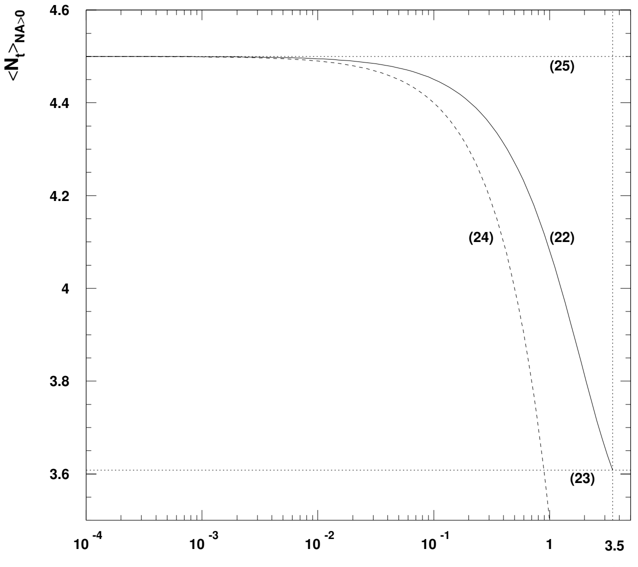

Figure 1: $\langle N_t \rangle_{N_A \geq 1}$ for $\mathcal{L}/k = 3.6 \times 10^{29}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 3.5$) as a function of ν_A for the general case (22) and in the approximation for small ν_A (24).

Figure 2 shows the same for $\nu_t = 0.188$.

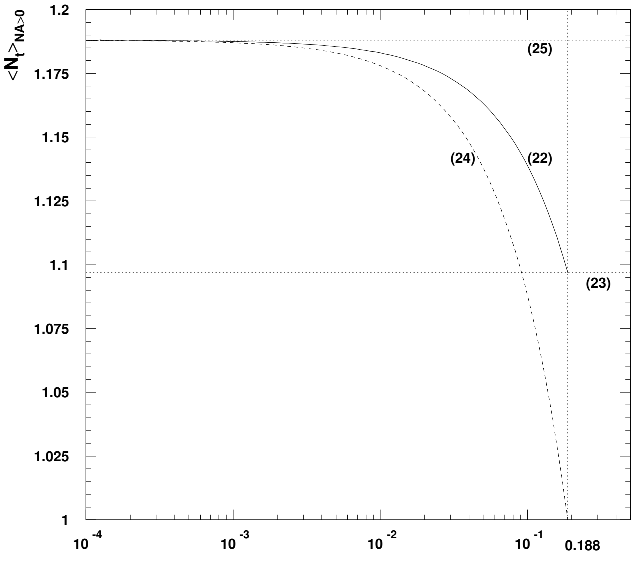

Figure 2: $\langle N_t \rangle_{N_A \geq 1}$ for $\mathcal{L}/k = 1.9 \times 10^{28}\,\mathrm{cm}^{-2}\mathrm{s}^{-1}$ ($\nu_t = 0.188$) as a function of ν_A for the general case (22) and in the approximation for small ν_A (24).

References

[1] Anonymous: private communication.